(solved by assuming $Pr = \frac{3}{4}$, $\mu \propto T^{1/2}$, perfect gas with constant specific heats and Stokes' relation between the viscosity coefficients) is sonic and is given by

$$\Delta(\alpha = \tfrac{1}{6}) = 0.260 + 0.9379\epsilon + 2.884\epsilon^2 + \ldots.$$

Finally, it may be noted that 1) for the effect of the corrections in the Rankine-Hugoniot relations on the surface conditions one has to calculate the third-order boundary layer and 2) following Ref. 3 the analysis can be extended to the case when the flow in the shock layer (including the boundary layer) is considered compressible.

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Derivatives of Eigenvalues and Eigenvectors

LYNN C. ROGERS*
Wright-Patterson Air Force Base, Ohio

Introduction

MANY times it is desirable or even mandatory to find the effects of a design parameter change on the dynamic stability and/or response characteristics of a system. This occurs, for example, when a design is unsatisfactory and improvement is sought on a cut and dry basis, sometimes guided by the most flimsy (but only available) logic. Effects predicted by incrementing the parameter and finding the new solution can be sensitive to inherent numerical difficulties as well as costly. Costs include use of highly skilled personnel when there are many severe demands for their time.

It is highly desirable to have available an accurate, efficient tool to compute directly and efficiently the effects of a design parameter change. Such a tool would receive wide application in finding the gradient of a dynamic type constraint variable as a function of the design parameters in an automated optimization procedure. It would also be valuable in less sophisticated design studies. It has the highly significant advantage of contributing to the physical understanding and insight of a problem.

The homogeneous part of a set of arbitrary order, linear, constant coefficient differential equations may be written in first-order matrix form

$$A\dot{y} + By = 0 \tag{1}$$

where the $N \times N$ matrices A and B are functions of the design parameters, and need not be symmetric, real, or hermitian. Previously published results are restricted to the symmetric case. Some of the y coordinates are the original displacement coordinates, some may be first derivatives of the original displacement coordinates, etc. Assuming the solution

$$y = \varphi e^{\alpha t} \tag{2}$$

leads to the eigen-problem

$$\alpha_i A\varphi_i + B\varphi_i = 0 \tag{3}$$

which has the characteristic equation

$$|\alpha A + B| = 0 \tag{4}$$

where α is an eigenvalue and φ is the corresponding eigenvector.

Consider the eigen-problem

$$\beta_j A^T\theta_j + B^T\theta_j = 0 \tag{5}$$

which has the characteristic equation

$$|\beta A^T + B^T| = |\beta A + B| = 0 \tag{6}$$

the same as (4). Equation (5) becomes

$$\alpha_j A^T\theta_j + B^T\theta_j = 0 \tag{7}$$

Premultiplying (3) by θ_j^T and (7) by φ_i^T, transposing the latter, and subtracting gives

$$(\alpha_i - \alpha_j)\theta_j^T A\varphi_i = 0 \tag{8}$$

The following orthogonality relations result:

$$\theta_j^T A\varphi_i = \theta_j^T B\varphi_i = 0,\ i \neq j \tag{9}$$

assuming distinct eigenvalues.

Apparently Traill-Nash[1] was the first to develop orthogonality relations for nonsymmetric matrices and apply them to dynamics. See also Halfman[2] and Foss.[3]

Since there are two sets of eigenvectors involved here, two normalization conditions must be imposed. It is convenient to normalize φ_i such that the element corresponding to a displacement coordinate with the largest modulus is set equal to unity, then to normalize θ_i such that

$$\theta_i^T A\varphi_i = 1 \tag{10}$$

Note that it is neither necessary to solve both the eigen-problems, nor to find all N solutions, because

$$\Theta_{n\times N}{}^T A_{N\times N}\Phi_{N\times n} = I_{n\times n},\ n \leq N$$

and

$$\Theta^T = \Phi^T[A\Phi\Phi^T]^{-1} \tag{11}$$

where Θ and Φ are matrices whose columns are the eigenvectors θ and φ, respectively.

Derivative of an Eigenvalue

Now start with the Rayleigh Quotient written as

$$\alpha_i\theta_i^T A\varphi_i + \theta_i^T B\varphi_i = 0 \tag{12}$$

and take the partial derivative with respect to a parameter ρ_k

$$\alpha_{i,k}\theta_i^T A\varphi_i + \alpha_i\theta_{i,k}{}^T A\varphi_i + \alpha_i\theta_i^T A_{,k}\varphi_i + \alpha_i\theta_i^T A\varphi_{i,k} + \theta_{i,k}{}^T B\varphi_i + \theta_i^T B_{,k}\varphi_i + \theta_i^T B\varphi_{i,k} = 0 \tag{13}$$

Collecting terms gives

$$\alpha_{i,k}\theta_i^T A\varphi_i + \theta_{i,k}{}^T\{\alpha_i A\varphi_i + B\varphi_i\} + \theta_i^T[\alpha_i A_{,k} + B_{,k}]\varphi_i + \lfloor\alpha_i\theta_i^T A + \theta_i^T B\rfloor\varphi_{i,k} = 0 \tag{14}$$

Received November 20, 1969.

* Aerospace Engineer, Design Criteria Branch, Structures Division, Air Force Flight Dynamics Laboratory; also Graduate Student, Engineering Mechanics Department, Ohio State University.

which, because of (3, 7, and 10), becomes

$$\alpha_{i,k} = -\theta_i^T[\alpha_i A_{,k} + B_{,k}]\varphi_i \quad (15)$$

Equation (15) is the desired expression for the derivative of an eigenvalue. An analogous expression for A and B symmetric, which results in φ and θ being identical (with proper normalization), was covered by Wittrick.[4] Obviously, an approximation to the eigenvalue after a set of parametric increments $\Delta\rho_k$ is

$$\alpha_i^* \simeq \alpha_i + \sum_k \alpha_{i,k}\Delta\rho_k \quad (16)$$

Somewhat as an aside, it is relevant to note that, in some eigen-problems, the angle in the complex plane between the imaginary axis and a line from the origin through the eigenvalue is of interest (e.g., in vibrations and dynamics, where the angle is indicative of damping factor). Depending on the relative magnitude of the real and imaginary parts of the derivative, the angle may or may not be increased as the real part of the eigenvalue is made more negative. Therefore, it may be necessary to consider the derivative of the quantity b_i, which occurs when the eigenvalue is written as

$$\alpha_i = -b_i w_i + j(1 - b_i^2)^{1/2} w_i \quad (17)$$

Taking the partial with respect to ρ_k, equating the resulting real and imaginary parts to the numerical expression, and solving for the derivative of the quantity b_i gives

$$b_{i,k} = -(1 - b_i^2)^{1/2}[(1 - b_i^2)^{1/2}(\alpha_{i,k})_R + b_i(\alpha_{i,k})_I]/w_i \quad (18)$$

Derivatives of an Eigenvector

Next, start with (3) and take the partial with respect to the parameter ρ_k

$$[\alpha_{i,k}A + \alpha_i A_{,k} + B_{,k}]\varphi_i + [\alpha_i A + B]\varphi_{i,k} = 0 \quad (19)$$

Because of completeness, the derivative of the eigenvector may be expressed as a linear combination of the eigenvectors

$$\varphi_{i,k} = \sum_j a_{ijk}\varphi_j \quad (20)$$

Substitute (20) into (19) and premultiply by θ_l^T

$$\theta_l^T[\alpha_{i,k}A + \alpha_i A_{,k} + B_{,k}]\varphi_i + \theta_l^T[\alpha_i A + B]\sum_j a_{ijk}\varphi_j = 0 \quad (21)$$

which, because of (9, 10, and 12), and when j is substituted for l, becomes

$$\alpha_{i,k}\delta_i + \theta_j^T[\alpha_i A_{,k} + B_{,k}]\varphi_i + a_{ijk}(\alpha_i - \alpha_j) = 0 \quad (22)$$

For $j \neq i$,

$$a_{ijk} = -\theta_j^T[\alpha_i A_{,k} + B_{,k}]\varphi_i/(\alpha_i - \alpha_j) \quad (23)$$

Note that (22), with $j = i$, is an alternate derivation for (15).

Denote the lth element in the ith eigenvector, which was set equal to 1, by φ_{li}. To be consistent, the vector φ_i must be normalized in the same way both before and after the increment in the parameter. This means that the same element must still be equal to 1

$$\varphi_{li} = 1,\ \varphi_{li,k} = \sum_j a_{ijk}\varphi_{lj} = 0 \quad (24)$$

and, therefore

$$a_{iik} = -\sum_{j,j\neq i} a_{ijk}\varphi_{lj} \quad (25)$$

Therefore (20, 23, and 25) give the expression for the derivative of an eigenvector. Analogous expressions for symmetric matrices were given by Fox and Kapoor.[5]

Summary

Expressions for the derivatives of eigenvalues and eigenvectors of a very general system have been derived. These results have vast potential in application to eigen-problems. By using the expressions, a set of increments in the design variables may be selected to yield the desired improvements in the system characteristics of interest. Applications include vibration, dynamics, flight control, etc.; i.e., any problem that is modeled by a set of arbitrary order, linear, constant coefficient differential equations possessing distinct eigenvalues. In particular, the eigenvalue derivative is seen to have profound possibilities in the automated optimum design of systems where dynamic response and/or dynamic stability are considerations.

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Simple Controllability Test for Systems of the Form $\ddot{X} = AX + BU$

Cathy Achareke* and G. J. Lastman†
University of Waterloo, Waterloo, Ontario, Canada

MANY physical systems are governed by a second-order differential equation of the form

$$\ddot{X}(t) = AX(t) + BU(t) \quad (1)$$

where $X(t)\ E^n$; $U(t)\ E^m$; $1 \leq m \leq n$; and A, B are constant $n \times n$ and $n \times m$ matrices, respectively. For example, a system of the form (1) could be obtained from Newton's Second Law of Motion. Frequently, the controllability of Eq. (1) is desired. The controllability test[1] requires a system to be given in the form

$$\dot{Z}(t) = CZ(t) + DU(t) \quad (2)$$

This requires that new variables $Y(t) = \dot{X}(t)$ be introduced, so that

$$Z(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} 2n\text{-vector} \quad (3)$$

$$C = \begin{bmatrix} \Theta_1 & I \\ A & \Theta_1 \end{bmatrix} 2n \times 2n \text{ matrix} \quad (4)$$

$$D = \begin{bmatrix} \Theta \\ B \end{bmatrix} 2n \times m \text{ matrix} \quad (5)$$

Received December 31, 1969. This work was supported in part by a grant from the Department of University Affairs, Province of Ontario.

* Graduate Student, Department of Applied Mathematics.
† Assistant Professor, Department of Applied Mathematics.